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The effect of time-dependent gravity with multiple frequencies on the thermal convective stability of a fluid layer

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Abstract

The effect of a time-dependent gravity vector with a double frequency on the thermal stability of a fluid layer is explained physically and by way of a calculation. The analogy of this problem to the time-dependent acceleration of the base plate or pivot of a simple linear pendulum is made clear. Specifically it is shown that if a fluid layer that is heated from below is unstable to a time-dependent gravity field for two different frequencies, it may regain stability if both frequencies act in concert with each other. © 2002 Elsevier Science Ltd. All rights reserved.

1. Background and physics

It is well known [1] that there is an onset of convection when a layer of fluid, erstwhile quiescent, is heated from below under a constant gravitational field, provided that the temperature difference exceeds a critical value. This happens because the density is higher in the cold upper region and the fluid arrangement is top heavy. The only stipulation for the convection to set in from an erstwhile quiescent state is that the gravitational vector be collinear to the imposed temperature gradient. Otherwise the fluid will be in constant motion no matter how small the temperature difference. This stipulation continues to be true even if the gravitational field is time-dependent [2]. The critical conditions that determine the marginal stability boundary between the quiescent and convective states depend upon the frequency and magnitude of the gravitational field or the 'g' vector [3,4]. In fact, in a fluid layer, which is heated from below, the arrangement gains stability for a given frequency for a certain amplitude of the periodic component of the g vector and loses the stability once again for very high amplitudes. This curious situation can be understood, better, by considering the physics of an unstable pendulum whose base plate is subjected to a time-dependent displacement in the direction of gravity. The analogy between the fluid problem and the pendulum problem will be brought out in this paper time and again. In this regard turn to Fig. 1.

Imagine a pendulum hanging upward from a pivot i.e., with the bob upward. Now let the pivot of this pendulum move only in the vertical direction in a time-periodic manner. As gravity acts downward, the pendulum is normally unstable. But the motion of the pivot can make this pendulum stable. To see why this is so, observe that small perturbations acting on the pendulum bob will cause it to accelerate downward. If the base plate is also made to accelerate downward fast enough one can imagine a regaining of stability with the pendulum bob now moving back upward with respect to an observer sitting on the base plate. However a very fast acceleration of the base plate can cause the pendulum bob to overshoot its equilibrium position at the top and regain its instability. In other words, in pure physical terms one can imagine that there is a limited range of amplitudes and frequencies of the base plate acceleration wherein the pendulum bob is stable. Like Gresho and Sani [2], we shall see later on that the equations that model fluid convection resemble the equation that models the pendulum-oscillatory base plate system. Consequently it is understandable that there must be a range of amplitudes and frequencies of the time-dependent part of the gravity vector for which the fluid system is stable.

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Nomenclature

a	wave number
FC	Floquet coefficient
$g(t)$	time-dependent gravitational field
g_{dc}	fixed component of gravitational field
g_{ac}	alternating component of gravitational field
l	length of a pendulum
L	height of fluid layer
P	pressure
Pr	Prandtl number ν/κ
r	ratio of two frequencies
Ra	thermal Rayleigh number $g\beta\Delta TL^3/\nu\kappa$
Ra_{ac}	thermal Rayleigh number due to the time-dependent gravitational field $g_{ac}\beta\Delta TL^3/\nu\kappa$
Ra_{dc}	thermal Rayleigh number due to the constant gravitational field $g_{dc}\beta\Delta TL^3/\nu\kappa$
t	time
T	temperature
$\overline{T}$	reference temperature
$\vec{V}$	velocity vector
$\overline{V}$	characteristic velocity

Greek symbols

τ	time constant
$\tau_{\kappa\nu}$	mechanical disturbance time constant
τ_{td}	thermal disturbance time constant
$\tau_{bouyancy}$	buoyancy time constant
ω	angular frequency
ω_0	angular frequency of a pendulum with length l
κ	thermal diffusivity
ν	kinematic viscosity
β_T	coefficient of thermal expansion
ρ	density of fluid
ρ_0	reference density
δ	transformed variable defined in the text
ϕ	transformed variable defined in the text
ε	relative magnitude of time-dependent gravitational field g_{ac}/g_{dc}
λ	transformed variable defined in the text

Superscript

$*$	scaled values

Subscripts

b	bottom plate
t	top plate
dc	fixed component
ac	changing component

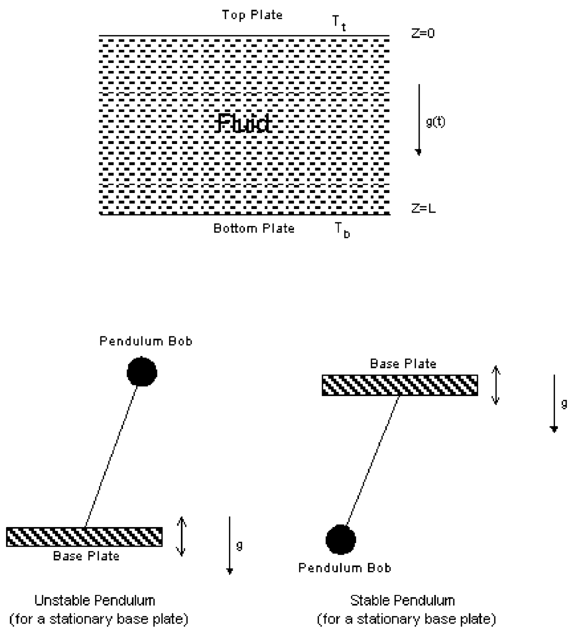

Fig. 1. Analogy between the stability of fluid layer, heated from below or top and confined between two parallel plate, and the stability of pendulum hanging upward or downward.

Now consider the reverse situation, i.e., the case of a stable pendulum. Here too, there is an analog to the fluid problem, this time however we imagine the fluid layer to be heated from above. If the pendulum is in its stable configuration, that is, a bob hanging below the base plate, a small frequency and amplitude of the base plate cannot really affect the stability of the arrangement. However if the amplitude was increased substantially for a given frequency one can imagine the bob amplitude increasing with time. One cannot foresee the possibility of the bob regaining stability for higher amplitudes of the plate motion unless perhaps a second frequency is imposed. This then takes us to the main theme of this paper.

The focus of this study is to extend the logic of the pendulum analog to the fluid convection problem where more than one frequency and amplitude can interact. Earlier studies show that at very high frequencies the gravity vector can be treated as nearly constant [5]. We do not make this assumption here. Instead the focus of this study is to analyze a model of fluid convection with a time-dependent gravitational field and to compare an analytical derivation with a numerical result for the case of a double frequency. We then show that a fluid layer that is heated from above may become unstable in a window of frequencies and amplitudes. In particular we show by a combination of analysis and computations that a system, that is individually unstable for two sets of amplitudes and frequencies, may actually become stable when the two sets act in concert with each other. To understand this better we move on to the model.

2. The model for the fluid layer

The model considered here is a layer of fluid confined between two horizontal parallel plates which are maintained at different but uniform temperatures. The entire arrangement is then subjected to a time-dependent gravitational field. Fig. 1 is a depiction of the physical problem being studied. The plates, the lower one at temperature T_b and the upper one at temperature T_t are assumed to be unbounded in the horizontal direction. The equations that model convection are written assuming that the density alone changes with temperature, its variation being of importance only where it occurs alongside an external acceleration. The equations of motions are then

$$\rho_0\frac{\partial\vec{V}}{\partial t}+\rho_0\left(\vec{V}\cdot\nabla\right)\vec{V}=-\nabla p+\mu\nabla^2\vec{V}+\rho_0\left(1+\frac{1}{\rho_0}\frac{\partial\rho}{\partial T}\bigg|_{\overline{T}}(T-\overline{T})\right)g(t)\vec{F} \tag{1}$$

where $\vec{F}=(0,0,1)$, $\vec{V}$, T and P are the solenoidal velocity, temperature and pressure fields respectively and ρ_0 and $\overline{T}$ are the reference density and temperature while $g(t)$ is the time-dependent gravitational field. The equation of energy is written assuming that viscous heat dissipation is unimportant. It takes the form

$$\frac{\partial T}{\partial t}+\vec{V}\cdot\nabla T=\kappa\nabla^2 T \tag{2}$$

The scaling of the equations depends largely on whether or not the dissipation effect of the kinematic viscosity exceeds that of thermal diffusivity. This then sets the scaling for velocity and time. The temperature scale on the other hand depends only on the temperature difference between the hot plate and the cold plate and is set such that the scaled temperature difference is unity, while the depth of the fluid, L, is chosen to be the length scale. Pretending that the kinematic viscosity, ν, is much less than the thermal diffusivity, κ, as typical of liquid metals, that is pretending that the Prandtl number is much less than unity, we get the characteristic velocity to be $\overline{V}=\nu/L$ and the characteristic time to be $\bar{t}=L^2/\nu$. This then yields the following scaled equation of motion

$$\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=-\nabla P+\nabla^2\vec{V}+\frac{Ra(t)}{Pr}T\vec{F} \tag{3}$$

The scaled energy equation is then

$$Pr\frac{\partial T}{\partial t}+Pr\left(\vec{V}\cdot\nabla T\right)=\nabla^2 T \tag{4}$$

Here the Prandtl number, Pr, is ν/κ and

$$Ra(t)=\frac{g(t)\beta\Delta TL^3}{\nu\kappa}\quad\text{where } \beta=-\frac{1}{\rho_0}\frac{\partial\rho}{\partial T}|_{\overline{T}}\text{ and}$$
$$\Delta T=T_b-T_t$$

Before going on it is worth reminding ourselves that $Ra(t)=Ra_{dc}+Ra_{ac}(t)$, where Ra_{dc} is the Rayleigh number based on the constant gravitational field while Ra_{ac} is the corresponding Rayleigh number for a time-dependent gravitational field. Moreover it is possible to view the Rayleigh number as the product of the ratio of time constants. If $\tau_{\kappa\nu}$, and τ_{td} are the time constants for the decay of mechanical and thermal disturbances and $\tau_{buoyancy}$ is the time constant for the decay of the buoyancy effect, then $Ra_{dc}=\tau_{\kappa\nu}\tau_{td}/\tau^2_{buoyancy}$, where

$$\tau_{\kappa\nu}=\frac{L^2}{\nu},\quad \tau_{td}=\frac{L^2}{\kappa}\text{ and }\tau^2_{buoyancy}=\frac{L}{g(-\frac{1}{\rho_0}\frac{\partial\rho}{\partial T}|_{\overline{T}}\Delta T)}$$

The modeling equations, which are nonlinear because of the interaction between $\vec{V}$ and T, are linearized about the quiescent state. One then assumes that $\vec{V}=\vec{V}_1\exp(ik_xx)\exp(ik_yy)$ and that $T=T_1\exp(ik_xx)\exp(ik_yy)$ with $k_x^2+k_y^2=a^2$ to get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2}-a^2\right)V_{z1}=\left(\frac{\partial^2}{\partial z^2}-a^2\right)^2V_{z1}-\frac{Ra(t)a^2}{Pr}T_1 \tag{5}$$

and

$$Pr\frac{\partial T_1}{\partial t}=PrV_{z1}+\left(\frac{\partial^2}{\partial z^2}-a^2\right)T_1 \tag{6}$$

Eliminating V_{z1}, the perturbed vertical component of velocity in favor of T_1 yields

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)\ddot{T}_1-\left(\frac{\partial^2}{\partial z^2}-a^2\right)^2\left(1+\frac{1}{Pr}\right)\dot{T}_1+\frac{1}{Pr}\left(\frac{\partial^2}{\partial z^2}-a^2\right)^3T_1+\frac{Ra(t)a^2}{Pr}T_1=0 \tag{7}$$

where the overdot represents a derivative with respect to time.

Taking the special case of stress free horizontal boundaries simplify the calculations henceforth and this only requires that $T_1=T_1(t)\sin(\pi z)$. The model then becomes

$$\ddot{T}_1(t)+\left(\pi^2+a^2\right)\left(1+\frac{1}{Pr}\right)\dot{T}_1(t)+\frac{(\pi^2+a^2)^2}{Pr}T_1(t)-\frac{Ra(t)a^2}{Pr(a^2+\pi^2)}T_1(t)=0 \tag{8}$$

This equation already resembles the equation that models a pendulum with frictional damping. On letting $T_1(t) = \phi(t)\exp(-1/2(\pi^2 + a^2)(1 + (1/Pr))t)$, the resulting equation viz.

$$\ddot{\phi} + \left[-\frac{(a^2+\pi^2)^2}{4}\left(1 - \frac{1}{Pr}\right)^2 - \frac{Ra(t)a^2}{Pr(a^2+\pi^2)} \right]\phi = 0 \tag{9}$$

assumes the form known as the Mathieu equation.

Let $Ra(t) = Ra_{dc}(1 + \varepsilon_1\cos(\omega_1^* t) + \varepsilon_2\cos(\omega_2^* t))$ where ω_1^* and ω_1^* are the dimensionless angular frequencies of the g field and ω's are the unscaled frequencies. Substituting this into the above along with $\omega_1^* t = 2\tau$ and $r = \omega_2^*/\omega_1^* = \omega_2/\omega_1$ yields

$$\ddot{\phi} + \left[-\frac{(a^2+\pi^2)^2 v^2}{\omega_1^2 L^4}\left(1 - \frac{1}{Pr}\right)^2 - \frac{4Ra_{dc}a^2v^2}{Pr(a^2+\pi^2)\omega_1^2L^4} \times (1 + \varepsilon_1\cos(2\tau) + \varepsilon_2\cos(2r\tau)) \right]\phi = 0 \tag{10}$$

The overdots now represent derivatives with respect to τ.

If $Pr \gg 1$, Eq. (10) assumes the form

$$\ddot{\phi} + \left[-\frac{(a^2+\pi^2)^2 \kappa^2}{\omega_1^2 L^4}(1 - Pr)^2 - \frac{4Ra_{dc}Pra^2\kappa^2}{(a^2+\pi^2)\omega_1^2L^4} \times (1 + \varepsilon_1\cos(2\tau) + \varepsilon_2\cos(2r\tau)) \right]\phi = 0 \tag{11}$$

In shorthand the model is

$$\ddot{\phi} + \left(\delta^2 + 2\lambda_1\cos(2\tau) + 2\lambda_2\cos(2r\tau)\right)\phi = 0 \tag{12}$$

where for low Pr number

$$\delta^2 = -\frac{(a^2+\pi^2)^2 v^2}{\omega_1^2 L^4}\left(1 - \frac{1}{Pr}\right)^2 - \frac{4Ra_{dc}a^2v^2}{Pr(a^2+\pi^2)\omega_1^2L^4} \tag{13}$$

$$\lambda_1 = -\frac{2Ra_{dc}a^2v^2\varepsilon_1}{Pr(a^2+\pi^2)\omega_1^2L^4} \text{ and } \lambda_2 = -\frac{2Ra_{dc}a^2v^2\varepsilon_2}{Pr(a^2+\pi^2)\omega_1^2L^4} \tag{14}$$

and for high Pr these constants become

$$\delta^2 = -\frac{(a^2+\pi^2)^2 \kappa^2}{\omega_1^2 L^4}(1 - Pr)^2 - \frac{4Ra_{dc}a^2\kappa^2Pr}{(a^2+\pi^2)\omega_1^2L^4} \tag{15}$$

$$\lambda_1 = -\frac{2Ra_{dc}Pra^2\kappa^2\varepsilon_1}{(a^2+\pi^2)\omega_1^2L^4} \text{ and } \lambda_2 = -\frac{2Ra_{dc}Pra^2\kappa^2\varepsilon_2}{(a^2+\pi^2)\omega_1^2L^4} \tag{16}$$

While these constants might appear to take large values when the Prandtl number is allowed to become very high or very low, a careful inspection of Eqs. (13)–(16) will reveal that as long as neither v/ω_1L^2 nor κ/ω_1L^2 is large, and this is a reasonable assumption, no difficulty will arise. Now several observations are worth making at this stage about this last second-order differential equation, i.e., Eq. (12).

First of all, if $\delta^2 > 0$, the problem corresponds to the heated from above case and looks like the model for the normal or stable pendulum where in that case $\delta^2 = \omega_0^2/\omega^2$ where $\omega_0 = \sqrt{g/l}$ with l being the length of the string that attaches the pendulum bob to the base plate. However if $\delta^2 < 0$ then two possibilities arise. Either the fluid can be heated from below or it may be heated from above, but not excessively. But if $Pr = 1$ then $\delta^2 < 0$ must correspond to the heated from below configuration only.

Second, if one lets $\lambda_2 = 0$ and $Pr = 1$ and in particular takes the case of $\delta^2 > 0$, then if $\delta^2 = 1, 4, 9\ldots$, a resonance is setup and ϕ becomes unbounded. This is a well established result [6]. Now it turns out that

$$\delta^2 = \frac{4a^2}{(a^2+\pi^2)}\left(\frac{\tau_{td}^2}{\tau_{\kappa v}^2}\right)\frac{\tau_{imposed}^2}{\tau_{buoyancy}^2} \quad \text{and as } Pr = 1,$$

$$\delta^2 = \frac{4a^2}{(a^2+\pi^2)}\frac{\tau_{imposed}^2}{\tau_{buoyancy}^2}$$

This is interesting by itself for we learn that as long as $Pr = 1$ it does not matter how kinematically viscous or thermally diffusive a fluid is for it will become unstable when δ^2 takes a resonant value n^2 with 'n' being a positive integer.

Third, if $\delta^2 > 0$, observe that as long as the thermal expansion coefficient, β, is fixed, and the frequency of oscillation of gravity is fixed, the fluid layer will become unstable at precisely the same temperature difference ΔT no matter how viscous or diffusive it might be provided that $Pr = 1$. Contrast this with the fluid layer heated from below for the case of a constant gravitational field. Here ΔT for instability will depend on the actual value of the thermal diffusivity and kinematic viscosity.

Fourth, note that the lowest value of δ^2 is unity for the heated from above problem to resonate. This then translates into $4a^2/(a^2 + \pi^2) = \tau_{buoyancy}^2/\tau_{imposed}^2$. In turn, this means that, as 'a' becomes larger, the imposed frequency of oscillation must be twice the 'buoyancy' frequency for resonance to occur. This is similar to requiring $\omega = 2\sqrt{g/l}$ for resonance in the case of a stable pendulum.

Finally, observe that when $Pr \neq 1$ and $Ra_{dc} < 0$ such that $\delta^2 < 0$ then $\delta^2(Pr \neq 1) < \delta^2(Pr = 1)$ and the second-order equation for ϕ acts like a pendulum with frictional damping, whereas the case of $Pr = 1$ gives the equation that resembles the undamped pendulum. It is our view that by putting $Pr \neq 1$, the smaller of the two effects, viscosity and thermal diffusivity, limit the effect of the other and ultimately cause faster decay of the disturbances than in the case where they are of equal magnitude.

To understand what happens when the time-dependent acceleration is composed of two different frequencies with two different amplitudes, we must analyze the second-order equation with a modification made on the time-dependent coefficients and it is to this task that we now turn.

3. The 'ϕ ' equation with a double frequency

The second-order equation that determines ϕ when the time-dependent acceleration is made up of two terms, one for each frequency is

$$\ddot{\phi} + \left(\delta^2 + 2\lambda_1 \cos(2\tau) + 2\lambda_2 \cos(2r\tau)\right)\phi = 0 \tag{17}$$

Here 'r' is the ratio of the two frequencies and need not be an integer. This equation called the Mathieu equation for two frequencies gives the behavior of ϕ with respect to δ i.e., behavior that we are most interested in. If we find that ϕ becomes singular for certain δ^2 then it must mean that T becomes singular. We already know that T is singular when $\delta^2 = 1, 4, \ldots$, for the single frequency case. This singularity cannot be corrected by merely multiplying ϕ by $\exp(-1/2(\pi^2 + a^2)(1 + (1/Pr))t)$. The reason for this is that the singularity is greater than any exponential order. We now ask whether these ideas carry over to the multiple frequency case or if anything else unusual occurs. To answer this question we turn to the case of $\delta^2 > 0$. Our interest resides in understanding how ϕ behaves as the frequency and magnitude of the acceleration change. An analytical derivation is advanced for this special case of $\delta^2 > 0$ so that it may serve as a check to a numerical calculation for the case of the fluid being heated from above. The same numerical calculation is then used to obtain the behavior of ϕ for the heated from below case for which we have no analogous analytical result.

To go on and investigate the behavior of ϕ, expand it and δ^2 in terms of λ_1 about $\delta^2 = d^2$ where d^2 is either n^2, n being an integer or where $d^2 = n^2 r^2$. As an example, suppose that $d^2 = n^2$ and $n = 2$ then

$$\phi(t; \lambda_1) = \phi_0(t) + \lambda_1 \phi_1(t) + \lambda_1^2 \phi_2(t) + \cdots \tag{18}$$

$$\delta^2 = 4 + \lambda_1 \delta_1 + \lambda_1^2 \delta_2 + \cdots \tag{19}$$

If λ_1 is related to λ_2 such that $\lambda_2 = m\lambda_1^2$ then the above expansion for ϕ and δ^2 when substituted into the Mathieu equation yields

$$\begin{aligned}&\ddot{\phi}_0 + \lambda_1 \ddot{\phi}_1 + \lambda_1^2 \ddot{\phi}_2 + \cdots + \left(4 + \lambda_1 \delta_1 + \lambda_1^2 \delta_2 + \cdots\right) \\ &\quad \times \left(\phi_0 + \lambda_1 \phi_1 + \lambda_1^2 \phi_2 + \cdots\right) + 2\lambda_1 \cos 2t \left(\phi_0 + \lambda_1 \phi_1 \right. \\ &\quad \left. + \lambda_1^2 \phi_2 + \cdots\right) + 2m\lambda_1^2 \cos 4t \left(\phi_0 + \lambda_1 \phi_1 + \lambda_1^2 \phi_2 + \cdots\right) = 0\end{aligned} \tag{20}$$

On equating the coefficients of like powers of λ_1 we obtain $\ddot{\phi}_0 + 4\phi_0 = 0$ which yields $\phi_0 = a\cos 2t + b\sin 2t$ and

$$\begin{aligned}\ddot{\phi}_1 + 4\phi_1 = &-\delta_1(a\cos 2t + b\sin 2t) - 2(a\cos 2t \\ &+ b\sin 2t)\cos 2t.\end{aligned}$$

As ϕ_1 must be bounded we see that $\delta_1 = 0$ and a particular solution for ϕ_1 is then

$$\phi_1 = -\frac{a}{4} + \frac{a}{12}\cos 4t + \frac{b}{12}\sin 4t. \tag{21}$$

Like wise

$$\begin{aligned}\ddot{\phi}_2 + 4\phi_2 = &-\delta_2(a\cos 2t + b\sin 2t) - \left[-\frac{a}{2}\cos 2t \right. \\ &+ \frac{a}{12}(\cos 6t + \cos 2t) + \frac{b}{12}(\sin 6t \\ &\left. + \sin 2t)\right] - m[a(\cos 6t + \cos 2t) \\ &+ b(\sin 6t - \sin 2t)]\end{aligned} \tag{22}$$

and the boundedness of ϕ_2 requires that

$$a\delta_2 - \frac{5a}{12} + ma = 0$$

and

$$b\delta_2 + \frac{b}{12} - mb = 0$$

from which either $a = 0$ or $\delta_2 = (5/12) - m$ and either $b = 0$ or $\delta_2 = -(1/12) + m$. Noting that both a and b cannot be simultaneously zero and assuming the first possibility leads to

$$\delta^2 = 4 - \tfrac{1}{12}\lambda_1^2 + m\lambda_1^2$$

Assuming the second possibility leads to

$$\delta^2 = 4 + \tfrac{5}{12}\lambda_1^2 - m\lambda_1^2$$

These two relationships for δ^2 and λ_1 are the loci along which the solution of the Mathieu equation i.e., ϕ, is periodic in nature and bounded in time. This is graphed for the particular case when $\lambda_1 = \lambda_2$ i.e., when $m = 1/\lambda_1$.

Notice that two curves depicted in Fig. 2 result. One curve results from

$$\delta^2 = 4 - \tfrac{1}{12}\lambda_1^2 + \lambda_1$$

while the other comes out of

$$\delta^2 = 4 + \tfrac{5}{12}\lambda_1^2 - \lambda_1$$

Now, we may wonder whether there is any real restriction by assuming that $\lambda_2 = m\lambda_1^2$. In fact one could very well have written $\lambda_2 = m\lambda_1$ and proceeded with the calculation much as above and then let $m = 1$ to obtain a new result. In such a case the λ_1 and δ^2 curve would still

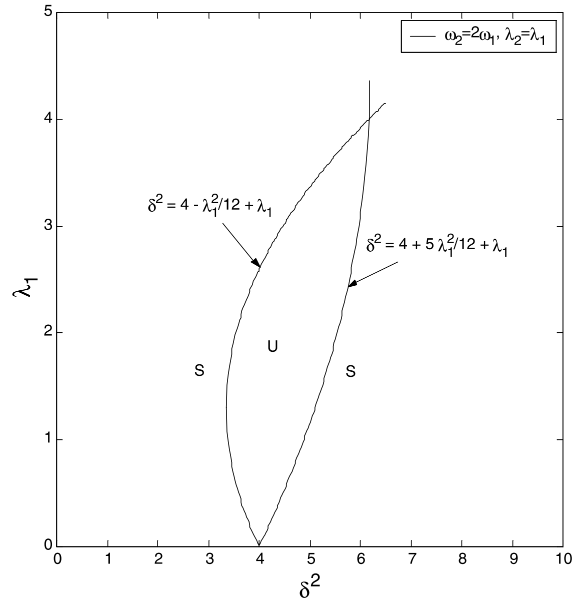

Fig. 2. Analytical solution of Mathieu solution for $\delta^2 = 4$. The inside region of the curve represents the stable solution while outside region has unstable solution.

look much like Fig. 2. In fact while the accuracy of the form is unknown, the actual relationship between λ_1 and δ^2 turns out to be invariant to the relationship assumed between λ_1 and λ_2 provided that λ_1 and λ_2 are both small. As mentioned earlier, the purpose of providing the solution of ϕ by the expansion method shown above, is to offer a check on a numerical calculation that will be presented. This numerical solution also gives the relationship between λ_1 and δ^2 such that ϕ does not become unbounded. It tells us where $\phi(t)$ is stable, where it is unstable and how it decays or grows with time. It is to the task of obtaining a numerical solution that we now turn.

The method that gives us this information is that of Floquet coefficients [6]. In this method, the stability of the solutions to the Mathieu equation, ϕ, are obtained by introducing a Floquet coefficient, γ, such that ϕ is proportional to $\exp(\gamma\tau)$ i.e., $\exp\left(\gamma\omega_1^* t/2\right)$. Note here that ω_1^* and t are the scaled angular frequency and time respectively. Thus if γ is positive the solution, ϕ, to the Mathieu equation is unstable and neutral stability is obtained only if γ is zero. However our model is concerned with the temperature perturbation T_1 and just because ϕ is unstable and grows, it does not mean that T_1 is unstable and grows. In fact recall for $Pr \ll 1$ that

$$T_1(t) = \phi(t)\exp\left(-\frac{1}{2}(\pi^2 + a^2)\left(1 + \frac{1}{Pr}\right)t\right).$$

This motivates us to introduce a modified Floquet coefficient hereafter called FC and given by

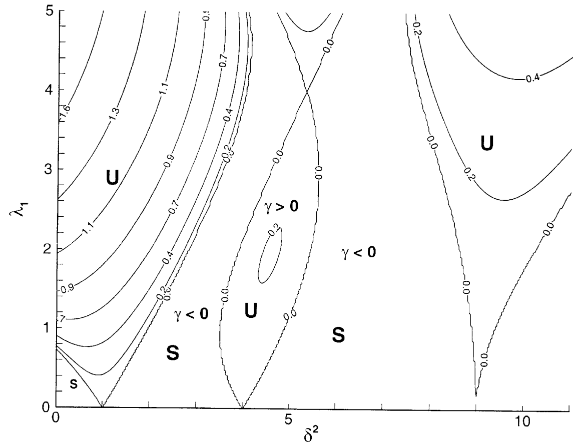

Fig. 3. Depiction of the stability curves for λ_1 vs. δ^2 with γ as a parameter. γ is obtained by solving the Mathieu equation numerically through the method of FC.

$$\mathrm{FC} = \gamma\frac{\omega_1 L^2}{2\nu} - \frac{1}{2}\left(a^2 + \pi^2\right)\left(1 + \frac{1}{Pr}\right) \quad \text{for } Pr \ll 1 \qquad (23)$$

and

$$\mathrm{FC} = \gamma\frac{\omega_1 L^2}{2\kappa} - \frac{1}{2}\left(a^2 + \pi^2\right)(1 + Pr) \quad \text{for } Pr \gg 1 \qquad (24)$$

Once again note that ω_1 is an unscaled angular frequency. If $\mathrm{FC} > 0$ the instability of the temperature equation is guaranteed. If $\mathrm{FC} < 0$ the temperature equation is conditionally stable to infinitesimal disturbances. Fig. 3 is a depiction of the stability curves where λ_1 is graphed against δ^2 with γ as a parameter. Given a fluid depth and physical properties and frequency of gravity one can compute δ^2 and λ_1 and thereafter obtaining γ from Fig. 3. In turn this gives us FC and tells us whether T_1 is stable or not. For regions of positive δ^2, Fig. 3 also can be compared with the analytical result plotted in Fig. 2 and one immediately sees that the comparison is favorable in the region of positive δ^2. This gives credence to the numerical result. In all our calculations we found γ to be real. Consequently the mode of oscillation at the neutral stability is always equal to the imposed frequency. To glean more from the calculations requires us to present some cases where the effect of Prandtl and Rayleigh numbers are seen on the stability of the fluid flow.

4. Discussion

Two sets of calculations have been performed and will be presented here. The first is for the case of the fluid layer heated from above and the second is for the situation where the fluid layer is heated from below.

The calculations presented here, assume the spatial wave number to be $\pi/\sqrt{2}$. This is, in fact, the critical wave number [1] when Ra is $27\pi^4/4$ and the sidewalls are stress free with the top and bottom boundaries conducting. The calculations were done for different Rayleigh numbers and Prandtl numbers and in so doing the value of L was chosen to be 5 cm, with the values of ν and κ corresponding to liquid tin ($Pr = 0.08$) and water ($Pr = 7.0$). The kinematic viscosity and thermal diffusivity for liquid tin are $\nu = 0.24 \times 10^{-6}\ \mathrm{m^2/s}$ and $\kappa = 3.0 \times 10^{-6}\ \mathrm{m^2/s}$ while for water they have been assumed to be $\nu = 1.0 \times 10^{-6}\ \mathrm{m^2/s}$ and $\kappa = 0.14 \times 10^{-6}\ \mathrm{m^2/s}$. It may be observed that when $Ra = -1000$ and $Pr = 0.08$, the g_{dc} is $13.1\mu g$ ($1\mu g = 9.8 \times 10^{-6}\ \mathrm{m^2/s}$) with a temperature drop of 15.2 °C whereas when $Pr = 7$, the g_{dc} is $46.0\mu g$. This observation is interesting in the light of the fact that the gravity level aboard a low gravity orbiter such as the Shuttle is of this order of magnitude.

4.1. Case 1: Heated from top

When the fluid is heated from above Ra is less than zero and δ^2 may be positive or negative. Consider now only the case when δ^2 is positive. Going back to the pendulum analog, this corresponds to the situation of the bob hanging downward from the base plate and is therefore inherently stable when the base plate is stationary. Now as Ra is decreased, the constant gravity case becomes more and more stable. To see what happens as a time-dependent gravitational field is imposed calculations are presented for the case of $Pr = 0.08$ and $Ra = -1000$ and -2000 respectively. Observe then from Fig. 4 that as Ra decreases so too does the region of stability which is the reverse of what we would see in the constant gravity case. This is a reasonable expectation for when the gravity vector is reversed, the more negative Ra causes more instability. In other words as δ^2 becomes larger the region of stability becomes smaller. And, this should not surprise us if we consider the pendulum analog. In that case $\delta^2 = 4g/l\omega^2$ where l is the length between the pivot and the bob.

A larger g or shorter l does indeed make the motion unstable. Fig. 4 also shows the stability region for a $Pr = 7$ and Ra equal to -1000 and -2000.0 respectively. Observe that the stability region increases by increasing the value of Pr for a given frequency. This is also understandable given that an increase in Prandtl number implies an increase in the kinematic viscosity and going

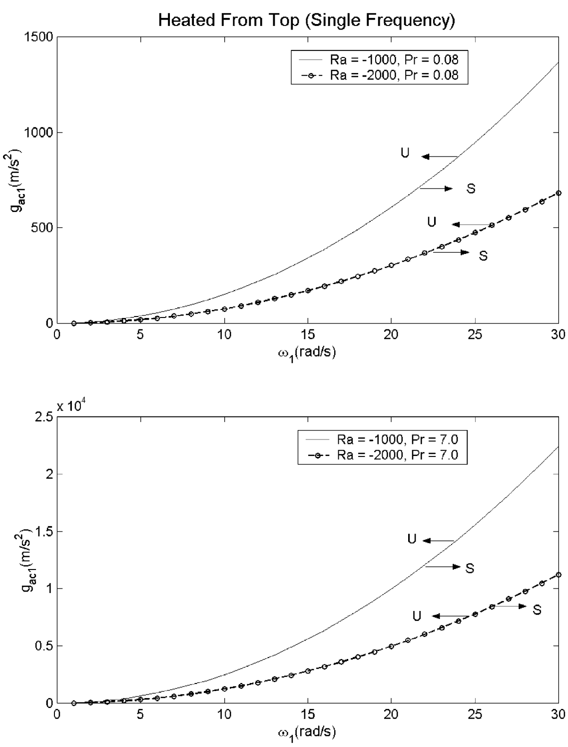

Fig. 4. Stability curve for single frequency gravity modulation with $Pr = 0.08$, 7 and $Ra = -1000$ and -2000. The fluid layer is heated from the top.

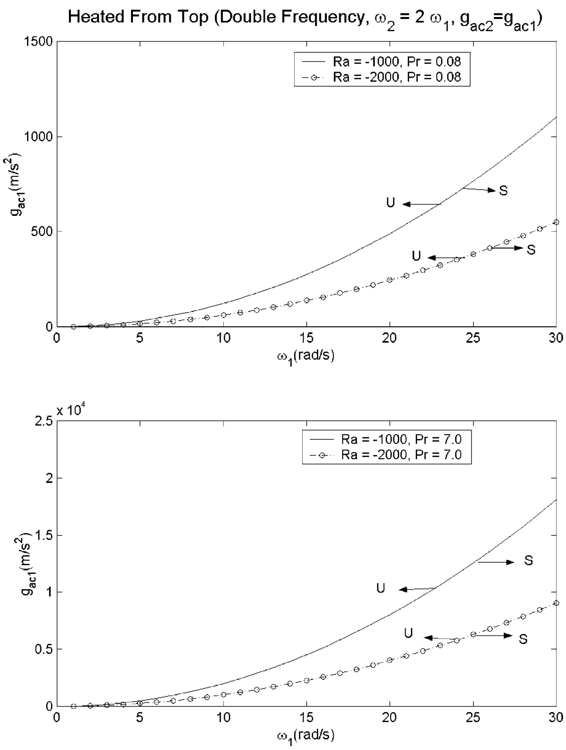

Fig. 5. Stability curves for two frequency gravity modulation with $Pr = 0.08$, 7 and $Ra = -1000$ and -2000 respectively. The fluid layer is heated from the top.

back to the formula for δ^2 when Pr is high we see that δ^2 becomes less positive as v increases. Increasing Pr when Pr is high therefore implies that mechanical perturbations die out quickly. The double frequency calculations were done assuming that $g_{ac1} = g_{ac2} = g_{ac}$ and that $\omega_2 = 2\omega_1$. Fig. 5 shows the effect of the double frequency for $Pr = 0.08$ and $Pr = 7$ when $Ra = -1000$ and -2000 respectively. What you now observe is that the region of stability has *decreased*. Here too the problem has an analogy to the stable pendulum. A second frequency, in that case, decreases the region of stability as well. An increase in Prandtl number once again causes a decrease in the stability region and is manifested by an increase in g_{ac1}, all of this showing the effect of the kinematic viscosity in delaying the instability.

We now move to the second case, that is the situation of the fluid heated from below and show some startling differences.

4.2. Case 2: Heated from bottom

Calculations were done for Rayleigh numbers (Ra_{dc}) of 657.4, 800, 1000, and 2000 for a variety of Prandtl numbers. In our calculations we have fixed the wave number to be $\pi/\sqrt{2}$. The values of g_{dc} were fixed at $13.1\mu g$ and $46.0\mu g$ ($1\mu g = 9.8 \times 10^{-6}$ m^2/s) for $Pr = 0.08$ and 6.94 respectively. These values were chosen based upon the real time gravity environment data acquired from a past US space shuttle mission viz., STS 87. A Rayleigh number of 657.4 ($< 27\pi^4/4$) corresponds to a stable situation when the steady case is considered. By introducing oscillations with a single frequency the layer is made unstable but only at high amplitudes. Contrast this with the case of $Ra = 800$ as seen in Fig. 6. This corresponds to an unstable situation in the steady case and indeed remains unstable for small amplitudes when a time-dependent g is imposed. This is also seen for the cases of $Ra = 1000$ and 2000 as shown in Figs. 7 and 8. Note that much like the pendulum analog when the amplitude of motion is increased, stability is obtained whereas for very high amplitudes instability is regained. Once again an increase in Prandtl number shows an increase in the stability regions even though this increase is very nominal. Moving onto the two frequencies case, Figs. 9 and 10 are the stability diagram for $Ra = 1000$ and 2000 and $Pr = 0.08$ and 7. In Fig. 9 $g_{ac2} = 0.2$ and $\omega_2 = 15$ while g_{ac1} is computed for different values of ω_1 such that the neutral stability point is reached. The value

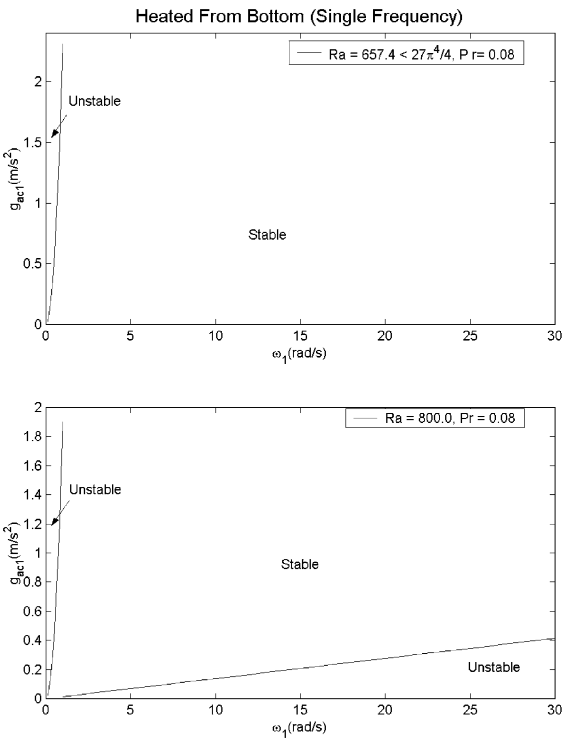

Fig. 6. Stability curves for single frequency gravity modulation with $Pr = 0.08$ and $Ra = 657.4$, 800. The fluid layer is heated from the bottom.

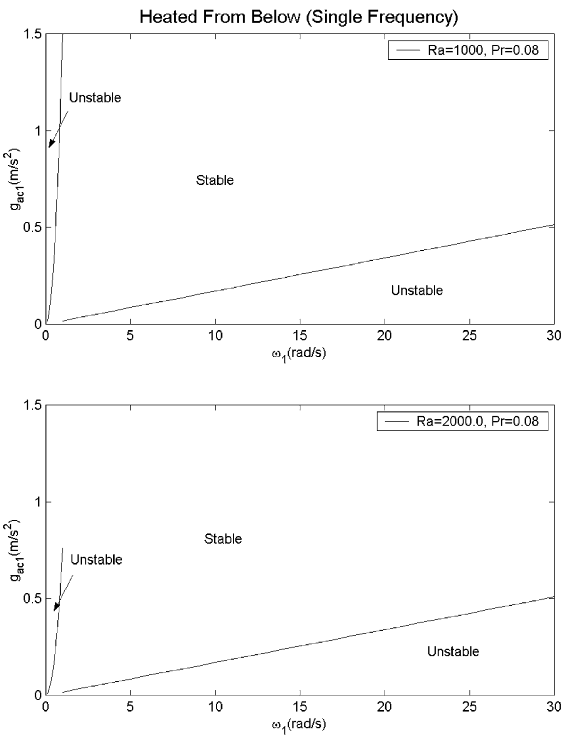

Fig. 7. Stability curves for single frequency gravity modulation with $Pr = 0.08$ and $Ra = 1000$, 2000. The fluid layer is heated from the bottom.

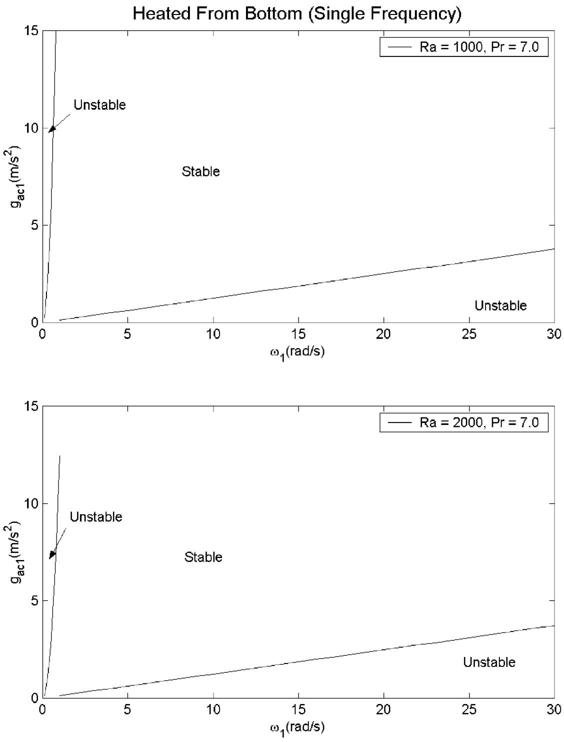

Fig. 8. Stability curves for single frequency gravity modulation with $Pr = 7.0$ and $Ra = 1000$, 2000. The fluid layer is heated from bottom.

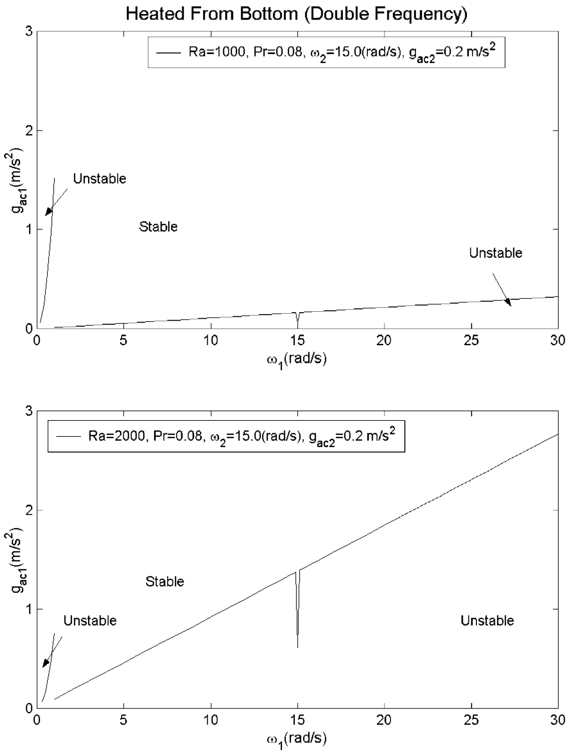

Fig. 9. Stability curves for two frequency gravity modulation with $Pr = 0.08$ and $Ra = 1000$, 2000. The g_{ac2} was fixed to 0.2 m/s^2 at the cyclic frequency of 15 rounds per second. The fluid layer is heated from bottom.

of $g_{ac2} = 0.2$ and $\omega_2 = 15$ corresponds to a point in the unstable region for a single frequency case and it is the purpose of the figure to show how the addition of a second frequency affected the stability when one of the frequencies and the accompanying amplitudes of the modulation are inherently unstable. The dominant feature that stands out is that the stability region increases substantially by introducing a second frequency. This unusual result means that while the system can be unstable to the individual frequencies it can be stable when both act in concert with each other. Notice in fact when $\omega_1 = 15$ the frequencies coincide and the stability changes sharply. This result should not be surprising when the pendulum analog is reconsidered. If an unstable pendulum gains stability at a certain amplitude and frequency then the introduction of a second frequency can actually reinforce the stability.

5. Conclusions and closing comments

In this study we have analyzed the thermal stability of a fluid layer subject to a double frequency time-dependent gravity field both analytically as well as by way of a calculation. Regardless of whether the fluid layer is heated from above or below we show that the effect of an increasing Prandtl number is to increase the region of stability thereby showing that the problem is mechanically driven. The fluid layer becomes unstable as it is heated from above with larger temperature gradients and this result is justified once the analogy with the simple pendulum is made clear. We also find that the fluid layer when heated from below becomes more stable when a second frequency is added to the first even though it might have been unstable to each frequency on its own.

Before we close, it is worthwhile giving an application to this study. An example where this work would be useful is in the determination of oxygen diffusivity in liquid tin by electrochemical titration [7]. Diffusion measurements involve concentration gradients and often generate convective instabilities that interfere with these measurements. Low-gravity experiments would be a way to reduce convective effects but even in outer space the gravity level is not constant but time dependent. It is therefore necessary to see the effect of

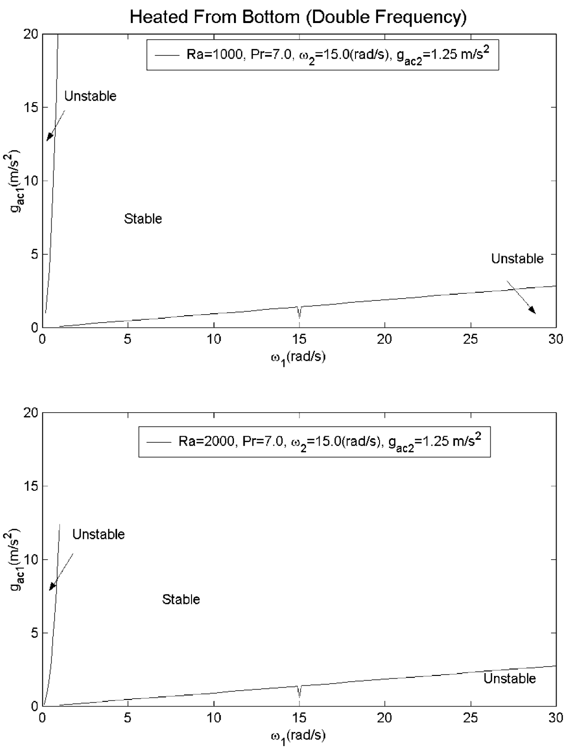

Fig. 10. Stability curves for two frequency gravity modulation with $Pr = 7.0$ and $Ra = 1000$, 2000. The g_{ac2} was fixed to 1.25 m/s^2 at the cyclic frequency of 15 rounds per second. The fluid layer is heated from bottom.

time-dependent gravity fields on the convective instability.

Acknowledgements

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